

# Cocycle Deformations of Algebraic Identities and $R$ -matrices

J. Scott Carter\*

University of South Alabama

Alissa S. Crans

Loyola Marymount University

Mohamed Elhamdadi

University of South Florida

Masahico Saito<sup>†</sup>

University of South Florida

February 15, 2008

## Abstract

For an arbitrary identity  $L = R$  between compositions of maps  $L$  and  $R$  on tensors of vector spaces  $V$ , a general construction of a 2-cocycle condition is given. These 2-cocycles correspond to those obtained in deformation theories of algebras. The construction is applied to a canceling pairings and copairings, with explicit examples with calculations. Relations to the Kauffman bracket and knot invariants are discussed.

## 1 Introduction

The 2-cocycle conditions of Hochschild cohomology of algebras and bialgebras are interpreted in deformations of algebras [12]. In other words, a map satisfying the associativity condition can be deformed to obtain a new associative map in a larger vector space using 2-cocycles.

Our motivation for this paper and other recent work [5, 6, 7] comes from the fact that quandle cocycles [3, 4] can be regarded as giving cocycle deformations of  $R$ -matrices (solutions to the Yang-Baxter equation (YBE)). Thus it was natural to ask if this principle could be applied to other algebraic constructions of  $R$ -matrices, to construct new  $R$ -matrices from old via 2-cocycle deformations. We have had some success in constructing new  $R$ -matrices from old using 2-cocycle deformations. Specifically, in [5], self-distributivity was revisited from the point of view of coalgebra categories, thereby unifying Lie algebras and quandles in these categories. Cohomology theories of Lie algebras and quandles were given a unified definition, and deformations of  $R$ -matrices were constructed. In [6], the adjoint map of Hopf algebras, which corresponds to the group conjugation map, was studied from the same viewpoint. A cohomology theory was constructed based on equalities satisfied by the adjoint map that are sufficient for it to satisfy the YBE. Finally, in [7] we presented an analog for Frobenius algebras using multiplication and comultiplication.

In the first half of this paper, we will describe a general principle of constructing deformation 2-cocycles from algebraic identities (such as associativity) that relate two (apparently) distinct tensor operators. Then in the second half, the principle is applied to the Kauffman bracket pairings to construct 2-cocycle deformations of bracket  $R$ -matrices. This is the same approach that we took in

---

\*Supported in part by NSF Grant DMS #0603926.

<sup>†</sup>Supported in part by NSF Grant DMS #0603876.

[5, 6, 7]. In this way, we are extending these methods to another algebraic structure. The cup-cap pairings are among the most simple of which we can conceive. Yet that there is a deformation theory even here strikes us as interesting. In our final section, we will discuss the knot invariants that result from these deformed  $R$ -matrices.

## 2 A general construction of 2-differentials

In the deformation theory of algebras and coalgebras [12, 17], deformation cocycles arise as the primary obstructions to extending a formal deformation of the algebraic structure. We observe in this section that the cocycle conditions, in general, can be loosely described as “infiltrating an algebraic condition” with an arbitrary function. A standard example will help illustrate the idea. Given a 2-cocycle  $\phi$ , the 2-cocycle condition of an associative algebra  $A$  is written as

$$\phi(ab \otimes c) + \phi(a \otimes b)c = \phi(a \otimes bc) + a\phi(b \otimes c),$$

for  $a, b, c \in A$  or in the multiplicative notation of linear maps  $\mu : A \otimes A \rightarrow A$

$$\phi(\mu \otimes 1) + \mu(\phi \otimes 1) = \phi(1 \otimes \mu) + \mu(1 \otimes \phi). \quad (1)$$

To derive this formula, take the associative law  $((ab)c) = (a(bc))$ , write it as  $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)$ , and put distinct subscripts on the multiplication maps for both sides of the identity to obtain  $\mu_1(\mu_2 \otimes 1) = \mu_1(1 \otimes \mu_2)$ . Then take a formal sum of each side by replacing each of the maps in turn by a map  $\phi$  to obtain

$$\phi(\mu_2 \otimes 1) + \mu_1(\phi \otimes 1) = \phi(1 \otimes \mu_2) + \mu_1(1 \otimes \phi).$$

By removing the subscripts, we obtain the 2-cocycle condition (1). Thus we see this scheme as a cocycle  $\phi$  infiltrating a formal sum of the identity.

This scenario has been generalized to a large variety of cases including our work in [5, 6, 7]. Our purpose in defining such generalizations is so that we can develop topological invariants of knots, manifolds, and knotted surfaces from cocycle conditions. The axiomatizations that we have developed are given via diagrammatic formulations. The diagrammatic versions often lead directly to topological interpretations via equivalences such as the Pachner moves, the Yang-Baxter condition, or the tetrahedral condition for knotted surfaces. On the other hand, the generality in which infiltration gives chain complexes is much more broad than the applications that we have found. Here, we will describe the situation in a broad setting, discuss the situations for which we have found interesting results, and point to some future generalities.

### 2.1 Single-term identities

For a linear map  $F : V^{\otimes p} \rightarrow V^{\otimes q}$ , we call  $(|_a \otimes F \otimes |_b) : V^{\otimes a+b+p} \rightarrow V^{\otimes a+b+q}$  a map *expanded from  $F$  by identities*. Here  $a, b, p$ , and  $q$  are non-negative integers, and the symbol  $|_x$  denotes the identity map on  $x$  tensor factors of the underlying vector space  $V$ . Let  $\text{Perm} : V^{\otimes p} \rightarrow V^{\otimes p}$  be a composition of maps expanded from the transposition by identities (i.e., a permutation of tensor factors, possibly the identity), called simply a *permutation* on  $V^{\otimes p}$ . A map written as

$(|_a \otimes F \otimes |_b) \circ \text{Perm} : V^{\otimes a+b+p} \rightarrow V^{\otimes a+b+q}$  is called a *map expanded from  $F$  by identities and transpositions*, where Perm is a permutation on  $V^{\otimes a+b+p}$ .

For a finite set of linear maps  $\mathcal{F} = \{F^\ell : \ell = 1, \dots, k\}$ , let  $L = R$  be an equation of linear maps from  $V^{\otimes p}$  to  $V^{\otimes q}$  such that both  $L$  and  $R$  are composites of maps expanded from  $F^\ell$ s ( $F^\ell \in \mathcal{F}$ ) and by identities and transpositions. We call this equation a *single-term identity*. In Fig. 1 (1), our diagrammatic convention is depicted. Each vertical string represents a tensor factor of  $V$ , and the diagram is read from bottom to top, so that the composition  $fg$  of maps is represented by the diagram of  $f$  on top of that of  $g$ . In (2), a multiplication map  $\mu : V \otimes V \rightarrow V$  is depicted on the left, and the single-term identity for associativity is depicted on the right. Similarly, (2) through (5) depict the corresponding maps and identities for the examples that follow.

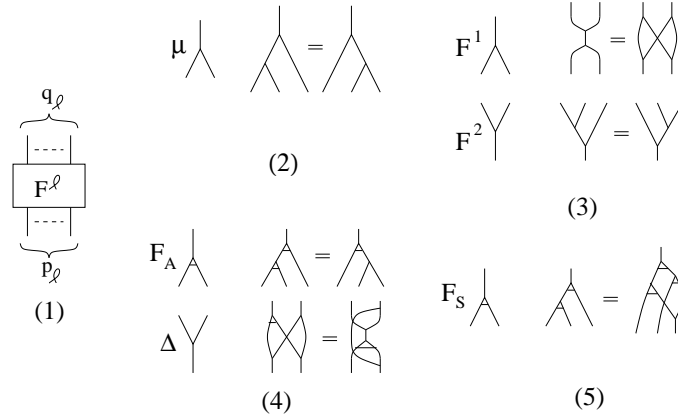


Figure 1: Single-term identities

**Example 2.1** Suppose that  $V$  is a vector space. Let  $F^1 : V \otimes V \rightarrow V$  and let  $F^2 : V \rightarrow V \otimes V$ . The *(single-term) bialgebra identities* are

$$\begin{aligned} F^1(F^1 \otimes |) &= F^1(| \otimes F^1), \\ (| \otimes F^2)(F^2) &= (F^2 \otimes |)F^2, \\ F^2 F^1 &= (F^1 \otimes F^1)(| \otimes X \otimes |)(F^2 \otimes F^2) \end{aligned}$$

where  $|$  denotes the identity and  $X$  denotes a transposition that acts, here, on the middle two tensorands. These correspond to associativity, coassociativity, and compatibility, which are illustrated in (3) above.

**Example 2.2** Let  $V$  be a Hopf algebra with multiplication  $\mu : V \otimes V \rightarrow V$ , comultiplication  $\Delta : V \rightarrow V \otimes V$ , and antipode  $S : V \rightarrow V$ . Let  $F_A : V \otimes V \rightarrow V$  denote the adjoint map. Then  $F_A = \mu(| \otimes \mu)(S \otimes |_2)(X \otimes |)(| \otimes \Delta)$  where, as before,  $X$  denotes a transposition, and  $|$  denotes the identity map. The *(single-term) adjoint identities* consist of the identities

$$\begin{aligned} F_A(F_A \otimes |) &= F_A(| \otimes \mu), \\ (F_A \otimes \mu)(| \otimes X \otimes |)(\Delta \otimes \Delta) &= (| \otimes \mu)(X \otimes |)(| \otimes \Delta)(| \otimes F_A)(X \otimes |)(| \otimes \Delta), \end{aligned}$$

which are illustrated in (4) above.

**Example 2.3** Suppose a vector space  $V$  has a cocommutative comultiplication  $\Delta : V \rightarrow V \otimes V$ . Then  $V$  has *(single-term) categorical self-distributivity* if there is a map  $F_s : V \otimes V \rightarrow V$  that satisfies:

$$F_s(F_s \otimes |) = F_s(F_s \otimes F_s)(| \otimes X \otimes |)(| \otimes | \otimes \Delta),$$

which is illustrated in (5) above. In [5], we also assumed that  $\Delta$  satisfied coassociativity (see Example 2.1).

**Example 2.4** Let  $V$  denote a vector space,  $\beta : V \otimes V \rightarrow \mathbb{K}$  denote a pairing, and  $\gamma : \mathbb{K} \rightarrow V \otimes V$  denote a copairing. Then the *(single-term) switchback identities* are

$$(| \otimes \beta)(\gamma \otimes |) = |, \quad \text{and} \quad (\beta \otimes |)(| \otimes \gamma) = |,$$

which are illustrated in Fig. 2.

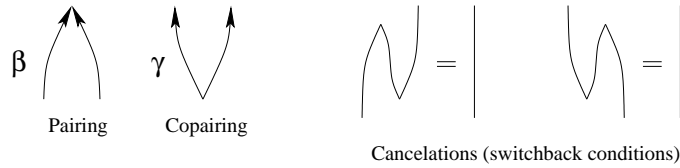


Figure 2: Diagrams for a pairing, a copairing and their identities

## 2.2 Elaborate plans – variable-distinctions of single-term equalities

Let  $L = R$  be a single-term identity. Let  $m(\ell)$  be the number of copies of  $F^\ell$  that appear in  $L$ , and write  $\mathcal{F}_L = \{F_j^\ell \mid \ell = 1, \dots, k; j = 1, \dots, m(\ell)\}$ , which we call *the distinguished variable set of  $L$* . Let  $\mathcal{L}(\mathcal{F}_L)$  be the formal expression obtained from  $L$  by replacing all  $F^\ell$ s by  $F_j^\ell$ s with distinct  $j$ . For example, for the associativity axiom of an algebra,  $\mu(\mu \otimes |) = \mu(| \otimes \mu)$ ,  $\mathcal{F} = \{\mu\}$ ,  $L = \mu(\mu \otimes |)$ ,  $\mathcal{F}_L = \{\mu_1, \mu_2\}$ , and  $\mathcal{L}(\mu_1, \mu_2) = \mu_1(\mu_2 \otimes |)$ . The same notation applies to the RHS, and again considering the associativity axiom we have  $R = \mu(| \otimes \mu)$ , and  $\mathcal{R}(\mu_1, \mu_2) = \mu_1(| \otimes \mu_2)$ .

**Definition 2.5** The formal equality  $\mathcal{L}(\mathcal{F}_L) = \mathcal{R}(\mathcal{F}_R)$  is called the *elaborate plan of the single-term identity  $L = R$* .

## 2.3 Infiltrations of elaborate plans – 2-cocycle conditions

To simplify the notation for substitution of a map  $f = f(x_1, \dots, x_n)$  with variables  $x_i$ ,  $i = 1, \dots, n$ , we use the notation  $f(x_i = g)$  to indicate that only a single variable  $x_i$  is substituted by  $g$ . That is,  $f(x_i = g) = f(x_1, \dots, x_{i-1}, g, x_{i+1}, \dots, x_n)$ . If multiple variables are substituted, the set of substitution rules are indicated, as in  $f(x_i = g, x_j = h) = f(x_1, \dots, g, \dots, h, \dots, x_n)$ , where  $i < j$  is assumed in this case.

**Definition 2.6** Let  $L = R$  be a single-term identity among linear maps  $\mathcal{F} = \{F^1, \dots, F^k\}$ . Let  $\mathcal{L}(\mathcal{F}_L) = \mathcal{R}(\mathcal{F}_R)$  be an elaborate plan of  $L = R$ , where  $\mathcal{F}_L = \{F_i^\ell \mid \ell = 1, \dots, k; i = 1, \dots, m(\ell)\}$

and  $\mathcal{F}_R = \{F_j^\ell \mid \ell = 1, \dots, k; j = 1, \dots, n(\ell)\}$ . We also write it as  $\mathcal{L}(\{F_i^\ell \mid \ell = 1, \dots, k; i = 1, \dots, m(\ell)\}) = \mathcal{R}(\{F_j^\ell \mid \ell = 1, \dots, k; j = 1, \dots, n(\ell)\})$ . Let  $\{\phi^\ell : V^{\otimes p_\ell} \rightarrow V^{\otimes q_\ell} : \ell = 1, \dots, k\}$  denote a collection of linear maps. An *infiltration of the elaborate plan*  $\mathcal{L}(\mathcal{F}_L) = \mathcal{R}(\mathcal{F}_R)$  is the formal sum

$$\sum_{\ell, i} \mathcal{L}(F_i^\ell = \phi^\ell, F_{i'}^\ell = F^\ell(i' \neq i)) = \sum_{\ell, j} \mathcal{R}(F_j^\ell = \phi^\ell, F_{j'}^\ell = F^\ell(j' \neq j)).$$

Here the substitution is made as follows. For a fixed  $\ell$ , there are  $m(\ell)$  copies  $F_i^\ell$  of  $F^\ell$  in the LHS  $L$ . First  $F_1^\ell$  is replaced by  $\phi^\ell$ , and all the other  $F_{i'}^\ell, i' \neq 1$ , are replaced by the original variable  $F^\ell$ . Then the second term is formally added after replacing  $F_2^\ell$  by  $\phi^\ell$  and other  $F_{i'}^\ell, i' \neq 2$ , are replaced by  $F^\ell$ . This is repeated for all  $\ell$ .

**Definition 2.7** We define the *2-differential* by LHS–RHS of the infiltration of the elaborate plan:

$$d^2(\phi^\ell : \ell = 1, \dots, k) = \sum_{\ell, i} \mathcal{L}(F_i^\ell = \phi^\ell, F_{i'}^\ell = F^\ell(i' \neq i)) - \sum_{\ell, j} \mathcal{R}(F_j^\ell = \phi^\ell, F_{j'}^\ell = F^\ell(j' \neq j)).$$

If a set of more than one single term equalities is given, then we define a 2-differential for each equality. Thus if equalities  $\{L_r = R_r \mid r = 1, \dots, s\}$  are given, denote their elaborate plans by  $\{\mathcal{L}_r(\mathcal{F}_{L_r}) = \mathcal{R}_r(\mathcal{F}_{R_r}) \mid r = 1, \dots, s\}$ , and we define

$$d^{2,r}(\phi^\ell : \ell = 1, \dots, k) = \sum_{\ell, i} \mathcal{L}_r(F_i^\ell = \phi^\ell, F_{i'}^\ell = F^\ell(i' \neq i)) - \sum_{\ell, j} \mathcal{R}_r(F_j^\ell = \phi^\ell, F_{j'}^\ell = F^\ell(j' \neq j)).$$

In this notation, the letter  $r$  specifies the equality and  $\ell$  specifies a map. Subscripts of  $F$  represent distinguished copies of a map.

**Example 2.8** We infiltrate the three bialgebra identities of Example 2.1 by  $\phi^1$  and  $\phi^2$  to obtain

$$d^{2,1}(\phi^1, \phi^2) = \phi^1(F^1 \otimes |) + F^1(\phi^1 \otimes |) - \phi^1(| \otimes F^1) - F^1(| \otimes \phi^1)$$

$$d^{2,2}(\phi^1, \phi^2) = (| \otimes \phi^2)(F^2) + (| \otimes F^2)(\phi^2) - (\phi^2 \otimes |)F^2 - (F^2 \otimes |)\phi^2$$

and

$$\begin{aligned} d^{2,3}(\phi^1, \phi^2) &= \phi^2 F^1 + F^2 \phi^1 - (\phi^1 \otimes F^1)(| \otimes X \otimes |)(F^2 \otimes F^2) \\ &- (F^1 \otimes \phi^1)(| \otimes X \otimes |)(F^2 \otimes F^2) - (F^1 \otimes F^1)(| \otimes X \otimes |)(\phi^2 \otimes F^2) - (F^1 \otimes F^1)(| \otimes X \otimes |)(F^2 \otimes \phi^2). \end{aligned}$$

## 2.4 The first differentials and $d^2 d^1 = 0$

Let  $V$  denote a finite dimensional vector space over a field  $\mathbb{K}$ , and as before, for each  $\ell = 1, 2, \dots, k$ , let  $F^\ell : V^{\otimes p_\ell} \rightarrow V^{\otimes q_\ell}$  denote a linear map. Let  $f : V \rightarrow V$ .

**Definition 2.9** For each  $\ell$ , the *1-differential* is a map

$$d^{1,\ell} : \text{Hom}(V, V) \rightarrow \text{Hom}(V^{\otimes p_\ell}, V^{\otimes q_\ell})$$

defined by

$$d^{1,\ell}(f) = \sum_{i=1}^{q_\ell} (|_{i-1} \otimes f \otimes |_{q_\ell-i}) F^\ell - \sum_{j=1}^{p_\ell} F^\ell (|_{j-1} \otimes f \otimes |_{p_\ell-j}).$$

$$\mathbf{d}^{l,r}(\text{---}\bigcirc\text{---}) = \left( \begin{array}{c} \mathbf{1} \\ \text{---}\bigcirc\text{---} \\ \Sigma \\ \mathbf{i} \\ \text{---}\text{---}\text{---} \end{array} \left[ \begin{array}{c} \text{---}\text{---}\text{---} \\ \mathbf{F}^l \\ \text{---}\text{---}\text{---} \end{array} \right] \right) - \left( \begin{array}{c} \Sigma \\ \mathbf{j} \\ \text{---}\text{---}\text{---} \\ \text{---}\bigcirc\text{---} \\ \mathbf{j} \end{array} \left[ \begin{array}{c} \text{---}\text{---}\text{---} \\ \mathbf{F}^l \\ \text{---}\text{---}\text{---} \end{array} \right] \right)$$

Figure 3: The first differential  $d^{1,\ell}$

A diagrammatic representation of the 1-differential is depicted in Fig. 3. The proof of the following proposition, then, can be easily visualized by this diagram.

**Proposition 2.10** *Let  $L = R$  be a single-term identity among  $\{F^\ell \mid \ell = 1, \dots, k\}$ , and  $d^2$  be the 2-differential of  $L = R$  with variables  $\{\phi^\ell\}$ . Let  $d^{1,\ell}$  be the 1-differential for  $\{F^\ell \mid \ell = 1, \dots, k\}$ . Then for all  $\ell = 1, \dots, k$ ,*

$$d^2(d^{1,1}(f), \dots, d^{1,k}(f)) = 0.$$

*Proof.* Recall that each term of  $d^2(\phi^1, \dots, \phi^k)$  contains exactly one of  $\{\phi^1, \dots, \phi^k\}$ . After substituting  $\phi^\ell = d^{1,\ell}(f)$ , each term contains exactly one copy of  $f$ . Hence each term, after substitution, is written as  $L_1(|i \otimes f \otimes |j) L_2$  or  $R_1(|i \otimes f \otimes |j) R_2$  where  $L = L_1 L_2$ ,  $R = R_1 R_2$  are the LHS and RHS of the given single-term identity  $L = R$ . If neither  $L_1$  nor  $L_2$  is the identity map, then there are exactly two copies of the term  $L_1(|i \otimes f \otimes |j) L_2$  in  $d^2(d^{1,1}(f), \dots, d^{1,k}(f))$ : one from  $\sum_{i=1}^{q_\ell} (|_{i-1} \otimes f \otimes |_{q_\ell-i}) F^\ell$ , and the other from  $-\sum_{j=1}^{p_\ell} F^\ell (|_{j-1} \otimes f \otimes |_{p_\ell-j})$ , and they cancel. The same argument applies to the terms of the form  $R_1(|i \otimes f \otimes |j) R_2$ . After canceling, we are left with terms of the form  $(|i \otimes f \otimes |j) L$ ,  $L(|i \otimes f \otimes |j)$ ,  $(|i \otimes f \otimes |j) R$ , and  $R(|i \otimes f \otimes |j)$ . The original identity implies, then, that the terms  $(|i \otimes f \otimes |j) L$  and  $(|i \otimes f \otimes |j) R$  cancel, and so do the terms  $L(|i \otimes f \otimes |j)$  and  $R(|i \otimes f \otimes |j)$ .  $\square$

## 2.5 Deformations and 2-differentials

The general construction defined above is based on deformations of algebras, and we describe the relation for associative algebras following [17]. Let  $A$  be an associative algebra over  $\mathbb{K}$  with multiplication  $\mu : A \otimes A \rightarrow A$ . A deformation of  $(A, \mu)$  is the  $\mathbb{K}[[t]]$ -algebra  $(A_t, \mu_t)$ , where  $A_t = A \otimes \mathbb{K}[[t]]$  with the multiplication  $\mu_t = \mu + t\mu_1 + t^2\mu_2 + \cdots$ . Here the maps  $\mu_i : A \otimes A \rightarrow A$  are extended to  $A_t$ . The associativity of  $\mu_t$  implies the following two equations obtained from equating the coefficients of  $t$  and  $t^2$ , respectively:

$$\begin{aligned} \mu_1(\mu(a \otimes b) \otimes c) + \mu(\mu_1(a \otimes b) \otimes c) &= \mu_1(a \otimes \mu(b \otimes c)) + \mu(a \otimes \mu_1(b \otimes c)), \\ \mu_2(\mu(a \otimes b) \otimes c) + \mu_1(\mu_1(a \otimes b) \otimes c) + \mu(\mu_2(a \otimes b) \otimes c) \\ &= \mu_2(a \otimes \mu(b \otimes c)) + \mu_1(a \otimes \mu_1(b \otimes c)) + \mu(a \otimes \mu_2(b \otimes c)). \end{aligned}$$

The first equation is the Hochschild 2-cocycle condition for  $\mu_1$ , and this degree calculation explains the above general construction.

It is also known that the second equation, when written as

$$\begin{aligned}\psi(a \otimes b \otimes c) &= \mu_1(\mu_1(a \otimes b) \otimes c) - \mu_1(a \otimes \mu_1(b \otimes c)) \\ &= \mu(a \otimes \mu_2(b \otimes c)) - \mu_2(\mu(a \otimes b) \otimes c) - \mu(\mu_2(a \otimes b) \otimes c) + \mu_2(a \otimes \mu(b \otimes c)),\end{aligned}$$

implies that  $\mu_1(\mu_1(a \otimes b) \otimes c) - \mu_1(a \otimes \mu_1(b \otimes c))$  is a coboundary if  $\mu_t$  is associative up to degree 2. Moreover, one can check  $\psi(a \otimes b \otimes c) = \mu_1(\mu_1(a \otimes b) \otimes c) - \mu_1(a \otimes \mu_1(b \otimes c))$  is a 3-cocycle if  $\mu_1$  is a 2-cocycle.

The argument above with respect to the deformation cocycles for an associative algebra show explicitly that these are obtained by the infiltration theory. Our prior work on cohomology of self-distributive maps, the adjoint map in a Hopf algebra, and on Frobenius algebras also can be interpreted from this infiltration theory. Pedro Lopes pointed out to us that this idea works in great generality and the discussion above is a formulation of that idea. In the next section, we move to develop this idea in the case of switchback identities.

### 3 Cohomology of switchback pairs

#### 3.1 Preliminaries

It is known that a *bilinear form (or pairing)*  $\beta : V \otimes V \rightarrow \mathbb{K}$  on a vector space  $V$  over a field  $\mathbb{K}$  is *nondegenerate* if and only if there is a  $\gamma : \mathbb{K} \rightarrow V \otimes V$  such that  $(\beta \otimes 1)(1 \otimes \gamma) = 1$  and  $(1 \otimes \beta)(\gamma \otimes 1) = 1$ .

More generally, for a module  $V$  over a unital ring  $\mathcal{K}$ , we define a pair  $(\beta, \gamma)$ , where  $\beta \in \text{Hom}(V^{\otimes 2}, \mathcal{K})$  and  $\gamma \in \text{Hom}(\mathcal{K}, V^{\otimes 2})$ , to be a *switchback pair* on  $V$  over  $\mathcal{K}$  if they satisfy  $(\beta \otimes 1)(1 \otimes \gamma) = 1$  and  $(1 \otimes \beta)(\gamma \otimes 1) = 1$ . We call these conditions *switchback conditions*.

Our diagrammatic conventions representing the bilinear pairing  $\beta$ , copairing  $\gamma$ , and the above conditions are depicted in Fig. 2, from left to right, respectively. Parallel strings, representing tensor products of vector spaces, are read and oriented from bottom to top, when linear maps are applied. For a pairing, two strings merge at a single point, which we represent by a maximum with a corner, or a cusped maximum (instead of a smooth maximum). Similarly, a copairing is represented by a cornered minimum. Unless the orientations get confusing, it is always upward and is often abbreviated. These diagrammatic conventions have been used often in knot theory (see, for example, [14]).

**Example 3.1 Kauffman bracket pair.** Let  $V$  be a 2-dimensional vector space over  $\mathbb{C}$  with basis elements  $x$  and  $y$ , where  $\beta$  is defined on basis elements by:

$$\beta(x \otimes x) = 0, \quad \beta(x \otimes y) = iA, \quad \beta(y \otimes x) = -iA^{-1}, \quad \beta(y \otimes y) = 0,$$

where  $A$  is a variable. This is the famous pairing used for the Kauffman bracket [14]. The corresponding copairing is defined by

$$\gamma(1) = iA(x \otimes y) - iA^{-1}(y \otimes x).$$

### 3.2 Deformations by 2-cocycles

Following [17] that described deformations of bialgebras, we formulate a deformation of switchback pairs. Let  $(\beta, \gamma)$  be a switchback pair on a module  $V$  over a unital ring  $\mathcal{K}$ . A *deformation* of  $\mathcal{A} = (V, \beta, \gamma)$  is a triple  $\mathcal{A}_t = (V_t, \beta_t, \gamma_t)$  whose constituents are as follows: (1) The module  $V_t = V \otimes \mathcal{K}[[t]]$  is, as indicated, the tensor product of  $V$  with a formal power series. We make the identification  $V_t/(tV_t) \cong V$ . (2) The maps  $(\beta_t, \gamma_t)$  of  $(\beta, \gamma)$  are given by  $\beta_t = \beta + t\beta_1 + \cdots + t^n\beta_n + \cdots : V_t \otimes V_t \rightarrow \mathcal{K}$  and  $\gamma_t = \gamma + t\gamma_1 + \cdots + t^n\gamma_n + \cdots : \mathcal{K} \rightarrow V_t \otimes V_t$  where  $\beta_i : V \otimes V \rightarrow \mathcal{K}$  and  $\gamma_i : \mathcal{K} \rightarrow V \otimes V$ ,  $i = 1, 2, \dots$ , are sequences of pairings and copairings, respectively. Suppose  $\beta$  and  $\gamma$  satisfy the switchback conditions mod  $t$ , and suppose that there exist  $\beta_1 : V \otimes V \rightarrow \mathcal{K}$  and  $\gamma_1 : \mathcal{K} \rightarrow V \otimes V$  such that  $\beta + t\beta_1$  and  $\gamma + t\gamma_1$  satisfy the switchback conditions mod  $t^2$ . Define  $\xi_1, \xi_2 \in \text{Hom}(V^{\otimes 3}, \mathcal{K})$  by

$$\begin{aligned} (\beta \otimes 1)(1 \otimes \gamma) - 1 &= t\xi_1 \mod t^2, \\ (1 \otimes \beta)(\gamma \otimes 1) - 1 &= t\xi_2 \mod t^2, \end{aligned}$$

This situation describes the *primary obstructions* to formal deformations of switchback pairs to be the pair of maps  $(\xi_1, \xi_2)$ , as in [17].

For the switchback condition of  $\beta + t\beta_1$  and  $\gamma + t\gamma_1 \mod t^2$  we obtain:

$$\begin{aligned} ((\beta + t\beta_1) \otimes 1)(1 \otimes (\gamma + t\gamma_1)) - 1 &= 0 \mod t^2, \\ (1 \otimes (\beta + t\beta_1))((\gamma + t\gamma_1) \otimes 1) - 1 &= 0 \mod t^2, \end{aligned}$$

which is equivalent by degree calculations to:

$$\begin{aligned} (d^{2,1}(\beta_1, \gamma_1) =) & (\beta \otimes 1)(1 \otimes \gamma_1) + (\beta_1 \otimes 1)(1 \otimes \gamma) = \xi_1, \\ (d^{2,2}(\beta_1, \gamma_1) =) & (1 \otimes \beta)(\gamma_1 \otimes 1) + (1 \otimes \beta_1)(\gamma \otimes 1) = \xi_2. \end{aligned}$$

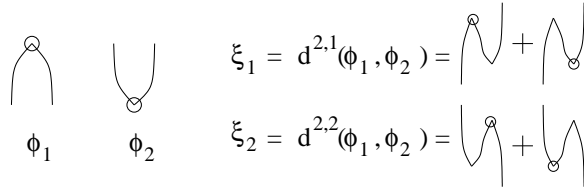


Figure 4: The 2-differentials

Thus we make the following definition:

**Definition 3.2** Let  $(\beta, \gamma)$  be a switchback pair on  $V$ . Let  $\phi_1 \in \text{Hom}(V^{\otimes 2}, \mathcal{K})$  and  $\phi_2 \in \text{Hom}(\mathcal{K}, V^{\otimes 2})$ . Then the *2-differentials* for  $(\phi_1, \phi_2)$  are defined by

$$\begin{aligned} d^{2,1}(\phi_1, \phi_2) &= (\beta \otimes 1)(1 \otimes \phi_2) + (\phi_1 \otimes 1)(1 \otimes \gamma), \\ d^{2,2}(\phi_1, \phi_2) &= (1 \otimes \beta)(\phi_2 \otimes 1) + (1 \otimes \phi_1)(\gamma \otimes 1). \end{aligned}$$

If  $(\phi_1, \phi_2)$  have vanishing 2-differentials, then they are called *2-cocycles*.



In Fig. 4, diagrammatic representations of  $(\phi_1, \phi_2)$  are depicted on the left, and the 2-differentials are depicted on the right. In summary we state the following:

**Proposition 3.3** (i) *The primary obstruction to deformation by  $(\beta_1, \gamma_1)$  of a switchback pair is  $(\xi_1, \xi_2)$  where  $\xi_1 = d^{2,1}(\beta_1, \gamma_1)$  and  $\xi_2 = d^{2,2}(\beta_1, \gamma_1)$ . Hence  $(\beta_1, \gamma_1)$  defines a deformation if and only if it forms a 2-cocycle.*

(ii) *If the primary obstruction vanishes,  $(\xi_1, \xi_2) = 0$ , (i.e.,  $(\beta_1, \gamma_1)$  are 2-cocycles), then the deformation  $(\tilde{\beta}, \tilde{\gamma}) = (\beta + t\beta_1, \gamma + t\gamma_1)$  is a switchback pair on  $V_t/(t^2V_t)$ .*

**Example 3.4** Let  $\mathcal{A} = (V, \beta, \gamma)$  be as in Example 3.1. Let  $\phi_1(a \otimes b) = \beta_{a,b}^1$  for basis elements  $\{a, b\} = \{x, y\}$ , and  $\phi_2(1) = \sum_{\{a,b\}=\{x,y\}} \gamma_1^{a,b}(a \otimes b)$ . Then the 2-cocycle conditions are formulated as:

$$\begin{aligned} d^{2,1}(\phi_1, \phi_2)(x) = 0 & : iA\beta_{x,x}^1y - iA^{-1}\beta_{x,y}^1x + iA\gamma_1^{y,x}x + iA\gamma_1^{y,y}y = 0, \\ d^{2,1}(\phi_1, \phi_2)(y) = 0 & : iA\beta_{y,x}^1y - iA^{-1}\beta_{y,y}^1x - iA^{-1}\gamma_1^{x,x}x - iA^{-1}\gamma_1^{x,y}y = 0, \\ d^{2,2}(\phi_1, \phi_2)(x) = 0 & : -iA^{-1}\gamma_1^{x,y}x - iA^{-1}\gamma_1^{y,y}y + iA\beta_{y,x}^1x - iA^{-1}\beta_{x,x}^1y = 0, \\ d^{2,2}(\phi_1, \phi_2)(y) = 0 & : iA\gamma_1^{x,x}x + iA\gamma_1^{y,x}y + iA\beta_{y,y}^1x - iA^{-1}\beta_{x,y}^1y = 0, \end{aligned}$$

which imply

$$\gamma_1^{y,y} = -\beta_{x,x}^1, \quad \gamma_1^{x,x} = -\beta_{y,y}^1, \quad \gamma_1^{y,x} = A^{-2}\beta_{x,y}^1, \quad \gamma_1^{x,y} = A^2\beta_{y,x}^1.$$

Hence in total there is a 4-dimensional solution space.

### 3.3 Cohomology groups

We discuss defining 1-differentials and 3-differentials, in relation to the above defined 2-differentials, and construct a chain complex in low dimensions.

**Definition 3.5** Let  $\mathcal{A} = (V, \beta, \gamma)$  where  $V$  is a module over a unital ring  $\mathcal{K}$ , and  $(\beta, \gamma)$  is a switchback pair. Define *chain groups* in low dimensions as follows:

$$\begin{aligned} C^1(\mathcal{A}) &= \text{Hom}(V, V), \\ C^2(\mathcal{A}) &= \text{Hom}(V^{\otimes 2}, \mathcal{K}) \oplus \text{Hom}(\mathcal{K}, V^{\otimes 2}), \\ C^3(\mathcal{A}) &= \text{Hom}(V, V)_{(1)} \oplus \text{Hom}(V, V)_{(2)}, \\ C^4(\mathcal{A}) &= \text{Hom}(V^{\otimes 2}, \mathcal{K}) \oplus \text{Hom}(\mathcal{K}, V^{\otimes 2}), \end{aligned}$$

where the subscripts for  $\text{Hom}(V, V)$  in  $C^3$  are to specify each factor. Define differentials as follows:

$$\begin{aligned} d^{1,1}(\eta) &= \beta(\eta \otimes 1) - \beta(1 \otimes \eta), & d^{1,2}(\eta) &= (\eta \otimes 1)\gamma - (1 \otimes \eta)\gamma, \\ d^{2,1}(\phi_1, \phi_2) &= (\beta \otimes 1)(1 \otimes \phi_2) + (\phi_1 \otimes 1)(1 \otimes \gamma), & d^{2,2}(\phi_1, \phi_2) &= (1 \otimes \beta)(\phi_2 \otimes 1) + (1 \otimes \phi_1)(\gamma \otimes 1), \\ d^{3,1}(\xi_1, \xi_2) &= \beta(\xi_1 \otimes 1) - \beta(1 \otimes \xi_2), & d^{3,2}(\xi_1, \xi_2) &= (\xi_2 \otimes 1)\gamma - (1 \otimes \xi_1)\gamma, \end{aligned}$$

where  $\xi_1 \in \text{Hom}(V, V)_{(1)}$  and  $\xi_2 \in \text{Hom}(V, V)_{(2)}$ . Then further define:

$$\begin{aligned} D_1(\eta) &= d^{1,1}(\eta) (\in \text{Hom}(V^{\otimes 2}, \mathcal{K})) + d^{1,2}(\eta) (\in \text{Hom}(\mathcal{K}, V^{\otimes 2})), \\ D_2(\phi_1, \phi_2) &= d^{2,1}(\phi_1, \phi_2) (\in \text{Hom}(V, V)_{(1)}) + d^{2,2}(\phi_1, \phi_2) (\in \text{Hom}(V, V)_{(2)}), \\ D_3(\xi_1, \xi_2) &= d^{3,1}(\xi_1, \xi_2) (\in \text{Hom}(V^{\otimes 2}, \mathcal{K})) + d^{3,2}(\xi_1, \xi_2) (\in \text{Hom}(\mathcal{K}, V^{\otimes 2})), \end{aligned}$$

and finally,

$$B^n(\mathcal{A}) = \text{Image}(D_{n-1}), \quad Z^n(\mathcal{A}) = \text{Ker}(D_n), \quad H^n(\mathcal{A}) = Z^n(\mathcal{A})/B^n(\mathcal{A}),$$

for appropriate values of  $n$ .

$$\eta = \text{---}\bigcirc\text{---} \quad d^{1,1}(\eta) = \text{---}\bigcirc\text{---}\diagup - \text{---}\diagdown\text{---}\bigcirc \quad d^{1,2}(\eta) = \text{---}\bigcirc\text{---}\diagdown - \text{---}\diagup\text{---}\bigcirc$$

Figure 5: The 1-differentials

Our diagrammatic conventions for representing cochains and differentials are as follows. A 1-cochain  $\eta \in \text{Hom}(V, V)$  is represented by a small white circle on a vertical string as depicted on the left of Fig. 5. The first differentials are depicted on the right of the figure.

$$\begin{array}{ccc} \text{---}\diagup & \xrightarrow{\xi_1} & \text{---}\diagdown \\ \text{---}\diagdown & & \text{---}\diagup \end{array}$$

Figure 6: Representing 3-cocycles

For a 3-cochain  $\xi_i \in \text{Hom}(V, V)_{(i)}$ ,  $i = 1, 2$ , there are two aspects of the diagrams depicted in Fig. 6. First, to distinguish elements in the two factors of  $\text{Hom}(V, V)$ , we use the graphs of  $y = \pm x^3$  with small white circles at the origin, respectively, as in the figure. When  $\xi_i$ ,  $i = 1, 2$ , is regarded as  $d^{2,i}(\phi_1, \phi_2)$ , respectively, the graphs  $y = \pm x^3$  are regarded as cusp points as in the figure. This is justified by the fact that the switchback condition, when regarded as a continuous move, corresponds to the cusp singularity of plane maps from a plane [11].

$$\begin{aligned} d^{2,1}D_1(\text{---}\bigcirc\text{---}) &= \text{---}\bigcirc\text{---}\diagup - \text{---}\diagdown\text{---}\bigcirc + \text{---}\diagup\text{---}\bigcirc - \text{---}\bigcirc\text{---}\diagdown = 0 \\ d^{2,2}D_1(\text{---}\bigcirc\text{---}) &= \text{---}\bigcirc\text{---}\diagdown - \text{---}\diagup\text{---}\bigcirc + \text{---}\diagdown\text{---}\bigcirc - \text{---}\bigcirc\text{---}\diagup = 0 \end{aligned}$$

Figure 7:  $D_2D_1 = 0$

**Theorem 3.6** *The above defined chain groups and differentials form a chain complex:*

$$0 \rightarrow C^1(\mathcal{A}) \xrightarrow{D_1} C^2(\mathcal{A}) \xrightarrow{D_2} C^3(\mathcal{A}) \xrightarrow{D_3} C^4(\mathcal{A}).$$

*Proof.* This follows from direct calculations using the switchback conditions, aided by diagrams. The fact  $D_2D_1 = 0$ , for example, is depicted in Fig. 7. For  $D_3D_2 = 0$ , diagrammatic calculations are

shown in Fig. 8. More specifically, on the left of the figure, we illustrate two ways to apply switchback conditions: left first or right first, starting from the “M” and “W” shaped curves, respectively. These moves correspond to cusps, and  $\xi_i$  ( $i = 1, 2$ ). Then these diagrams are substituted by linear combinations of other diagrams corresponding to  $d^{2,i}(\phi_1, \phi_2)$ , and the terms cancel as expected.  $\square$

$$\begin{aligned}
d^{3,1}D_2(\xi_1, \xi_2) &= \text{cusp} - \text{cusp} \\
&= \left( \text{M} + \text{M} \right) - \left( \text{M} + \text{M} \right) = 0 \\
d^{3,2}D_2(\xi_1, \xi_2) &= \text{cusp} - \text{cusp} \\
&= \left( \text{W} + \text{W} \right) - \left( \text{W} + \text{W} \right) = 0
\end{aligned}$$

Figure 8:  $D_3D_2 = 0$

**Example 3.7** For  $\mathcal{A} = (V, \beta, \gamma)$  as in Examples 3.1 and 3.4, we continue and compute cohomology groups. Let  $\eta \in C^1(\mathcal{A})$ , be written as  $\eta(a) = \sum_{b \in \{x, y\}} \eta_a^b \cdot b$ . Direct calculations show that  $D_1(\eta) = 0$  implies  $\eta_x^x = \eta_y^y$ , and  $\eta_x^y = \eta_y^x = 0$  unless  $A^2 + 1 = 0$ . This implies that  $Z^1(\mathcal{A}) = H^1(\mathcal{A}) \cong \mathbb{C}$  and  $B^2(\mathcal{A}) \cong \mathbb{C}^3$  unless  $A^2 + 1 = 0$ . Computations in Example 3.4 imply that  $Z^2(\mathcal{A}) \cong \mathbb{C}^4$  and  $B^3(\mathcal{A}) \cong \mathbb{C}^4$ , so that we obtain  $H^2(\mathcal{A}) \cong \mathbb{C}$ . Let  $\xi_i \in \text{Hom}(V, V)_{(i)} \subset C^3(\mathcal{A})$ ,  $i = 1, 2$ , be written as  $\xi_i(a) = \sum_{\{a, b\} = \{x, y\}} \xi_{i_a}^b(b)$ , then  $d^{3,1}(\xi_1, \xi_2) = 0$  implies  $\xi_{2_x}^x = \xi_{1_y}^y$ ,  $\xi_{2_y}^y = \xi_{1_x}^x$ ,  $\xi_{2_x}^y = -A^{-2}\xi_{1_x}^y$  and  $\xi_{2_y}^x = -A^2\xi_{1_y}^x$ . The second 3-differential  $d^{3,2}(\xi_1, \xi_2) = 0$  implies the same set of equations as the first. Hence we obtain  $Z^3(\mathcal{A}) \cong \mathbb{C}^4$  and  $H^3(\mathcal{A}) = 0$ .

**Remark 3.8** The degree 2 terms calculated in Section 2.5 for the Hochschild cohomology has the following analogue for switchback pairs. Let  $\beta_t = \beta + t\beta_1 + \cdots + t^n\beta_n + \cdots : V_t \otimes V_t \rightarrow \mathcal{K}$  and  $\gamma_t = \gamma + t\gamma_1 + \cdots + t^n\gamma_n + \cdots : \mathcal{K} \rightarrow V_t \otimes V_t$  be formal deformations, and assume that they also satisfy the switchback condition, which implies that the degree two terms satisfy

$$(\beta \otimes 1)(1 \otimes \gamma_2) + (\beta_1 \otimes 1)(1 \otimes \gamma_1) + (\beta_2 \otimes 1)(1 \otimes \gamma) = 0. \quad (2)$$

When we set  $\psi_1 = (\beta_1 \otimes 1)(1 \otimes \gamma_1)$ , and similarly  $\psi_2 = (1 \otimes \gamma_1)(\beta_1 \otimes 1)$ , we obtain two facts similar to the Hochschild case.

(1) If the switchback relation holds up to degree 2, then the above Equation (2) holds, and it implies that  $\psi$  is a coboundary:

$$\psi_1 = (\beta_1 \otimes 1)(1 \otimes \gamma_1) = -d^{2,1}(\beta_2, \gamma_2) = -(\beta \otimes 1)(1 \otimes \gamma_2) - (\beta_2 \otimes 1)(1 \otimes \gamma).$$

(2) The above  $\psi$  is a 3-cocycle:  $D_3(\psi_1, \psi_2) = 0$ . This is verified by direct calculations.

## 4 Deformations of $R$ -matrices by switchback pairs and Knot Invariants

### 4.1 Constructions and deformations of $R$ -matrices

In this section we present a construction of  $R$ -matrices from switchback pairs and their deformations by 2-cocycles.

$$\left\langle \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right\rangle_t = a \left\langle \begin{array}{c} \nwarrow \\ \nearrow \end{array} \right\rangle_t + b \left\langle \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right\rangle_t$$

Figure 9: The deformed  $R$ -matrix

**Lemma 4.1** *Let  $(\beta, \gamma)$  be a switchback pair on a  $\mathcal{K}$ -module  $V$  as above. Let  $\delta_0 = \beta\gamma(1) \in \mathcal{K}$ . Set  $R = a1 + b(\gamma\beta)$  for variables  $a$  and  $b$  taking values in  $\mathcal{K}$ . Then  $R$  is an invertible solution to the YBE if  $a$  and  $b$  are invertible and the equality  $a^2 + b^2 + \delta_0 ab = 0$  holds.*

*Proof.* This lemma is an oriented version of the Kauffman bracket, and seems folklore. Direct calculations after expanding the YBE by the skein relation and comparing the coefficients of corresponding terms gives the above equation. There are a few key computations of inverses, however, that we need to look at carefully later, and we mention these.

Set  $R' = a'1 + b'(\gamma\beta)$  for variables  $a'$  and  $b'$  and impose that  $R'$  is the inverse of  $R$  and satisfies the YBE. Then we obtain additional equations,  $a'^2 + b'^2 + \delta_0 a'b' = 0$  for  $R'$  to be a solution to the YBE, and  $aa' = 1$ ,  $ab' + a'b + \delta_0 bb' = 0$  for  $RR' = 1$ . If  $a$  is invertible, we set  $a' = a^{-1}$ , and the equation  $a^2 + b^2 + \delta_0 ab = 0$  gives  $\delta_0 b = -(a^2 + b^2)a^{-1}$ , and the equation  $ab' + a'b + \delta_0 bb' = 0$  becomes  $ab' + a^{-1}b - (a^2 + b^2)a^{-1}b' = 0$ , i.e.,  $b(1 - bb') = 0$ . With the substitutions  $a' = a^{-1}$  and  $b' = b^{-1}$  in  $a'^2 + b'^2 + \delta_0 a'b' = 0$ , we get back  $a^2 + b^2 + \delta_0 ab = 0$ . Hence the result follows. We note that the conditions are also necessary except the case  $b = 0$ , in which case we have rather trivial  $R$ -matrices  $R = a1$  and  $R^{-1} = a^{-1}1$ .  $\square$

The pair  $(\beta, \gamma)$  in Example 3.1, of course, gives the bracket, where  $a = b' = A$ ,  $a' = b = A^{-1}$ , and  $\delta_0 = -A^2 - A^{-2}$  in the lemma. Let  $(\tilde{\beta}, \tilde{\gamma}) = (\beta + t\beta_1, \gamma + t\gamma_1)$  be a deformation by 2-cocycles  $(\beta_1, \gamma_1)$ . By Proposition 3.3,  $(\tilde{\beta}, \tilde{\gamma})$  is a switchback pair on  $V_t/(t^2V_t)$  which is regarded as a module over  $\mathcal{K}[t]/(t^2)$ . The above lemma applies with  $R_t = a1 + b(\gamma + t\gamma_1)(\beta + t\beta_1)$ . The coefficients  $a, b$  and  $\delta_0$  needs to be recalculated, and by setting  $t = 0$ , we recover the original  $R$ -matrix. Using Example 3.4, we summarize this situation as follows for the Kauffman bracket.

**Proposition 4.2** *Let  $(\beta, \gamma)$  be a switchback pair for the Kauffman bracket on  $V$  over  $\mathcal{K}$  with the  $R$ -matrix defined by  $R = A1 + A^{-1}(\gamma\beta)$ . Let  $(\beta_1, \gamma_1)$  be 2-cocycles. Then the deformation  $R_t = a1 + b(\gamma + t\gamma_1)(\beta + t\beta_1)$  is a solution to the YBE if  $a^2 + b^2 + \delta_0 ab = 0$  and  $a, b$  are invertible, where*

$$\begin{aligned} \delta_0 &= (\beta + t\beta_1)(\gamma + t\gamma_1)(1) = (-A^2 - A^{-2}) + t(iA\beta_{x,y}^1 - iA^{-1}\beta_{y,x}^1) + t(iA\gamma_1^{x,y} - iA^{-1}\gamma_1^{y,x}) \\ &= (-A^2 - A^{-2}) + t [ i(A^2 - A^{-2})(A^{-1}\beta_{x,y}^1 + A\beta_{y,x}^1) ]. \end{aligned}$$

**Remark 4.3** A Temperley-Lieb algebra  $TL_n$  (see, for example, [14]) has generators  $e_i, i = 1, \dots, n$  for a positive integer  $n$ , with relations  $e_i e_{i+1} e_i = e_i$ ,  $e_{i+1} e_i e_{i+1} = e_{i+1}$  for  $i = 1, \dots, n-1$ ,  $e_i e_j = e_j e_i$  for  $|i - j| > 1$ , and  $e_i^2 = \delta e_i$  where  $\delta \in \mathbb{K}$  for the coefficient field  $\mathbb{K}$ . Graphically  $e_i$  is represented by a pair of cup and cap as for  $\gamma\beta$ .

It is well-known [ibid] that a switchback pair  $(\beta, \gamma)$  gives a representation of the  $TL_n$  by

$$e_i \mapsto 1^{\otimes(i-1)} \otimes (\gamma\beta) \otimes 1^{\otimes(n-i-1)},$$

where  $\delta = \delta_0 = \beta\gamma(1)$ . Thus the deformation of a switchback by 2-cocycles  $(\beta_1, \gamma_1)$  gives rise to a deformation of the representation by

$$e_i \mapsto 1^{\otimes(i-1)} \otimes (\gamma + t\gamma_1)(\beta + t\beta_1) \otimes 1^{\otimes(n-i-1)}$$

with  $\delta = \delta_0 = (\beta + t\beta_1)(\gamma + t\gamma_1)(1)$ .

**Remark 4.4** A closer inspection shows that, in fact, deformations of representations of the Temperley-Lieb algebra  $TL_n$  in the preceding remark can be obtained from a pair  $(\beta_1, \gamma_1)$  by

$$e_i \mapsto 1^{\otimes(i-1)} \otimes (\gamma + t\gamma_1)(\beta + t\beta_1) \otimes 1^{\otimes(n-i-1)}$$

if they satisfy  $d^{2,1}(\beta_1, \gamma_1) = d^{2,2}(\beta_1, \gamma_1)$ , which is derived from the relations

$$e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1} \quad \text{for } i = 1, \dots, n-1.$$

In the case of the Kauffman bracket pairings in Example 3.1, this condition is written as

$$\gamma_1^{x,x} = -\beta_{y,y}^1, \quad \gamma_1^{y,y} = -\beta_{x,x}^1, \quad \text{and} \quad \gamma_1^{x,y} - A^2 \beta_{y,x}^1 = A^2(\gamma_1^{y,x} - A^{-2} \beta_{x,y}^1).$$

Compare with Example 3.4.

## 4.2 Knot invariants from deformation 2-cocycles

For the rest of the section, we show that the cocycle deformations of the bracket give rise to evaluations of the Jones polynomial by truncated polynomials.

Let  $R$  be the  $R$ -matrix obtained by the skein relation in Lemma 4.1 and its deformation obtained in Proposition 4.2. We consider knot invariants obtained by Turaev's criteria [18]. For a map  $f : V \otimes V \rightarrow V \otimes V$ , let  $\text{Tr}_2(f) : V \rightarrow V$  denote the map obtained from  $f$  by taking the trace on the second tensor factor of  $V$ . The map  $\text{Tr}_2(f)$  is written as a composition of the coevaluation  $\text{coev} : \mathcal{K} \rightarrow V \otimes V^*$  and evaluation  $\text{ev} : V \otimes V^* \rightarrow \mathcal{K}$  maps by  $(1 \otimes \text{ev})f(1 \otimes \text{coev})$ , where  $V^*$  denotes the dual of  $V$ . These maps are defined for basis elements  $\{v_i | i = 1, \dots, n\}$  by  $\text{coev}(1) = \sum_{i=1}^n v_i \otimes v_i^*$  and  $\text{ev}(v_i \otimes v_j) = \delta(i, j)$ , where  $n$  is the dimension of  $V$ , and  $\delta(i, j)$  is Kronecker's delta. Diagrammatically,  $f$  is represented by a box with two strings at the top and bottom, and  $\text{Tr}_2(f)$  is represented by the diagram of  $f$  with its right top and right bottom strings connected by a small loop at its right. See the LHS of figures (2) and (3) in Fig. 10. In this case, the right-most string representing the dual space  $V^*$  is oriented downwards by convention, and the maps  $\text{coev}$  and  $\text{ev}$  are represented by smooth minimum and maximum, respectively, with orientation consistently going through the maximum and minimum, in contrast to the cusp maximum and minimum representing pairing and copairing, with colliding orientations.

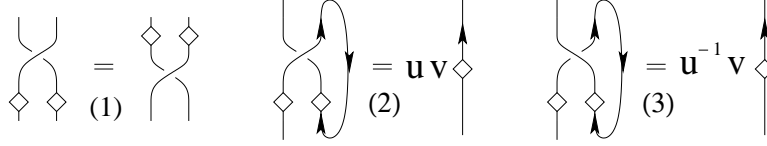


Figure 10: Turaev's conditions

**Theorem 4.5 (Turaev[18])** *Let  $R : V \otimes V \rightarrow V \otimes V$  be an (invertible) solution to the YBE on free module  $V$  over a commutative ring  $\mathcal{K}$  with unit. Suppose  $\nu : V \rightarrow V$  and  $u, v \in \mathcal{K}$  are invertible elements that satisfy (1)  $R \circ (\nu \otimes \nu) = (\nu \otimes \nu) \circ R$ , (2)  $\text{Tr}_2(R \circ (\nu \otimes \nu)) = uv\nu$ , and (3)  $\text{Tr}_2(R^{-1} \circ (\nu \otimes \nu)) = u^{-1}v\nu$ . Then these maps define a link invariant via the closed braid  $\hat{w}$  of an  $n$ -braid word  $w$  by*

$$T_R(\hat{w}) = u^{-\mathcal{W}(w)} v^{-n} \text{Tr}(\nu^{\otimes n} \circ R(w)),$$

where  $\mathcal{W}(w)$  denotes the writhe,  $\text{Tr}$  denotes the trace, and  $R(w)$  denotes the braid group representation induced from the  $R$ -matrix  $R$  on  $V^{\otimes n}$ .

The conditions are diagrammatically depicted in Fig. 10. The map  $\nu$  is represented by a small white rhombus.

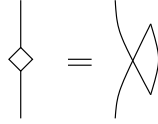


Figure 11: The map  $\nu$

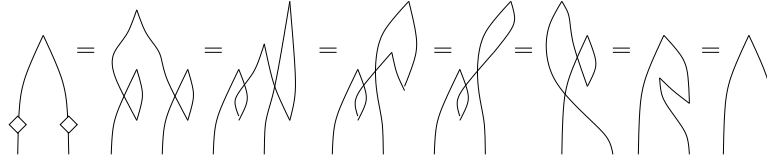


Figure 12: Lemma 4.6 (i)

To apply Turaev's construction for the  $R$ -matrix given in Lemma 4.1, we will need the following formulas.

**Lemma 4.6** *Let  $(\beta, \gamma)$  be a switchback pair on  $V$  over  $\mathcal{K}$ . Then for  $\nu = (1 \otimes \beta)(\tau \otimes 1)(1 \otimes \gamma)$ , the following hold: (i)  $\beta(\nu \otimes \nu) = \beta$ ,  $(\nu \otimes \nu)\gamma = \gamma$ . (ii)  $\text{Tr}_2((\gamma \otimes 1)(\beta \otimes 1)(1 \otimes \nu \otimes 1)) = 0$ .*

*Proof.* The diagram representing  $\nu$  is given in Fig. 11. For (i), Fig. 12 indicates a sketch of a proof of the first equality, and the vertical mirror images would represent a proof for the second. For (ii), Fig. 13 shows a proof.  $\square$

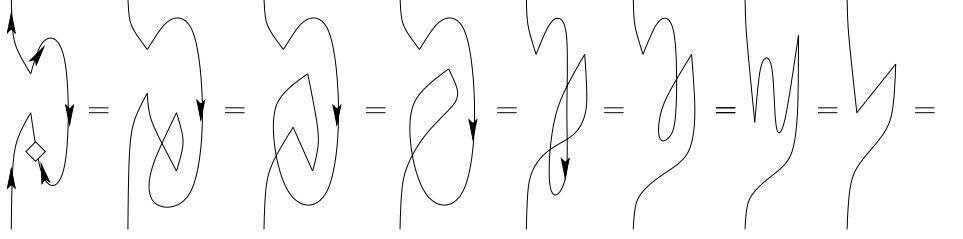


Figure 13: Lemma 4.6 (ii)

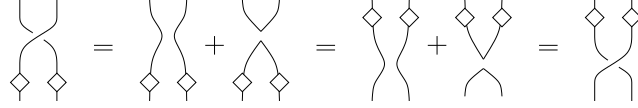


Figure 14: Condition (1)

**Proposition 4.7** *The  $R$ -matrix constructed in Lemma 4.1 from a pair  $(\beta, \gamma)$  defines a knot invariant by Turaev's criteria with  $\nu = (1 \otimes \beta)(\tau \otimes 1)(1 \otimes \gamma)$ .*

*Similarly, the deformed  $R$ -matrix  $R_t = a1 + b(\gamma + t\gamma_1)(\beta + t\beta_1)$  constructed in Proposition 4.2 defines a knot invariant by Turaev's criteria with*

$$\nu_t = (1 \otimes (\beta + t\beta_1))(\tau \otimes 1)(1 \otimes (\gamma + t\gamma_1)).$$

*Proof.* This follows from Theorem 4.5 by checking the three conditions. The first and the second are outlined in Figs. 14 and 15, respectively. The third is similar to the second.  $\square$

Let  $DB(K)$  denote the knot invariant defined by  $R_t = a1 + b(\tilde{\gamma}\tilde{\beta})$  and  $R_t^{-1} = a'1 + b'(\tilde{\gamma}\tilde{\beta})$  using  $\tilde{\beta} = \beta + t\beta_1$  and  $\tilde{\gamma} = \gamma + t\gamma_1$  as in Proposition 4.7, where  $(\beta_1, \gamma_1)$  are 2-cocycles. For the rest of the section, we compute the invariant  $DB(K)$ . From the equalities in Fig. 10, the formulas  $uv = \delta_0 a + b$  and  $u^{-1}v = \delta_0 a' + b'$  hold, where  $aa' = 1$ ,  $bb' = 1$  and  $a^2 + b^2 + \delta_0 ab = 0$ . See the proof of Lemma 4.1. The value of  $\delta_0$  is given in Proposition 4.2 as  $\delta_0$ . Multiplying the skein relations  $R_t = a1 + b(\tilde{\gamma}\tilde{\beta})$  and  $R_t^{-1} = a'1 + b'(\tilde{\gamma}\tilde{\beta})$  by  $u^{-1}$  and  $u$ , respectively, we obtain the skein relations  $DB(K_+) = au^{-1}DB(K_0) + bu^{-1}DB(K_\infty)$  and  $DB(K_-) = a'uDB(K_0) + b'uDB(K_\infty)$ , where  $DB(K_\infty)$  denotes the trace of a map with one crossing replaced with  $\tilde{\gamma}\tilde{\beta}$ . By eliminating the term  $DB(K_\infty)$ , we obtain the relation

$$(b'u)DB(K_+) - (bu^{-1})DB(K_-) = (ab' - a'b)DB(K_0).$$

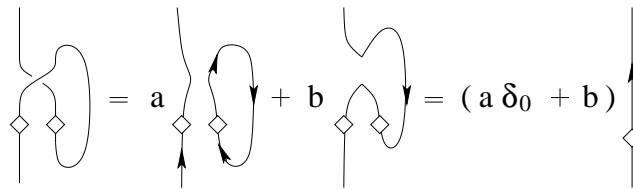


Figure 15: Condition (2)

Let  $c = a/b$ ,  $\ell = b^{-1}u$ , and  $m = ab' - a'b = c - c^{-1}$ . Then the skein relation for DB is the same as that of HOMFLYPT polynomial. We have  $\delta_0 = -(c + c^{-1})$ ,  $u = \ell b$ , and  $a = bc$ , and the relation for  $u^2$  is simplified as follows:

$$\begin{aligned} (a^{-1}\delta_0 + b^{-1})u &= (a\delta_0 + b)u^{-1}, \\ \ell - \ell^{-1} &= \delta_0(au^{-1} - a^{-1}u) \\ &= -(c + c^{-1})(c\ell^{-1} - c^{-1}\ell) \\ &= (-c^2\ell^{-1} + c^{-2}\ell) + (\ell - \ell^{-1}), \\ \ell^2 &= c^4, \quad \ell = \pm c^2. \end{aligned}$$

This gives the skein relation of the Jones polynomial up to sign. We summarize our calculations as follows.

**Proposition 4.8** *Let  $R_t$  be the cocycle deformation of the Kauffman bracket  $R$ -matrix defined in Proposition 4.2. Then the knot invariant DB defined from  $R_t$  as above is an evaluation of the Jones polynomial by a truncated polynomial.*

### 4.3 Conclusion

In this paper, we indicated that the construction of cohomology theories via deformations in low dimensions can be applied to broad classes of maps and identities in variety of algebraic structures. As an example, we presented such a construction for the Kauffman pairing and copairing, and carried out computations obtaining non-trivial cocycles. Thus the principle of cocycle deformations of  $R$ -matrices provides new solutions to the YBE. Probably due to the elegance of the Kauffman bracket and the rigidity of the Temperley-Lieb algebra (as pointed out to us by Vaughan Jones), the resulting knot invariants are evaluations of the Jones polynomial by truncated polynomials. Properties of the coefficients of the non-constant part, however, may still be of interest. The deformed  $R$ -matrices presented in this paper, as well as in [4, 5, 6, 7], and a general direction for cocycle deformation of identities suggested in this paper, indicate unifying relations between cocycle deformations of algebraic systems and invariants of low dimensional knots and manifolds.

The authors would like to thank the editors of this proceedings, Vaughan Jones, and Pedro Lopes for helpful conversations.

## References

- [1] Baez, J.C.; Crans, A.S., *Higher-Dimensional Algebra VI: Lie 2-Algebras*, Theory and Applications of Categories **12** (2004), 492–538.
- [2] Brieskorn, E., *Automorphic sets and singularities*, Contemporary math., 78 (1988), 45–115.
- [3] Carter, J.S.; Elhamdadi, M.; Saito, S., *Twisted Quandle homology theory and cocycle knot invariants*, Algebraic and Geometric Topology (2002) 95–135.
- [4] Carter, J.S.; Jelsovsky, D.; Kamada, S.; Langford, L.; Saito, M., *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. 355 (2003), no. 10, 3947–3989.



- [5] Carter, J.S.; Crans, A.; Elhamdadi, M.; Saito, S., *Cohomology of Categorical Self-Distributivity*, To appear in J. Homotopy Relat. Struc., Preprint, arXiv:math.GT/0607417.
- [6] Carter, J.S.; Crans, A.; Elhamdadi, M.; Saito, S., *Cohomology of the adjoint of Hopf algebras*, J. Gen. Lie Theory Appl., vol 2, No. 1 (2008) 19-34.
- [7] Carter, J.S.; Crans, A.; Elhamdadi, M.; Karadayi, E.; Saito, S., *Cohomology of Frobenius Algebras and the Yang-Baxter Equation*. Preprint, arXiv:0801.2567.
- [8] Crans, A.S., *Lie 2-algebras*, Ph.D. Dissertation, 2004, UC Riverside, available at arXiv:math.QA/0409602.
- [9] Eisermann, M., *Yang-Baxter deformations of quandles and racks*, Algebraic & Geometric Topology, 5 (2005), 537–562.
- [10] Fenn, R.; Rourke, C., *Racks and links in codimension two*, Journal of Knot Theory and Its Ramifications Vol. 1 No. 4 (1992), 343–406.
- [11] Golubitsky, M.; Guillemin, V. *Stable mappings and their singularities*, Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973. x+209 pp.
- [12] Gerstenharber, M; Schack, S.D., *Bialgebra cohomology, deformations, and quantum groups*, Proc. Nat. Acad. Sci. U.S.A., 87 (1990), 478–481.
- [13] Joyce, D., *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Alg., 23, 37–65.
- [14] Kauffman, L.H., *Knots and Physics*, World Scientific, Series on knots and everything, vol. 1, 1991.
- [15] Khovanov, M., *A categorification of the Jones polynomial*, Duke Math. J. **101(3)** (1999), 359–426.
- [16] Kock, J., *Frobenius algebras and 2D topological quantum field theories*, London Mathematical Society Student Texts (No. 59), Cambridge University Press, 2003.
- [17] Markl, M.; Stasheff, J.D., *Deformation theory via deviations*, J. Algebra 170 (1994), 122–155.
- [18] Turaev, V. G. *The Yang-Baxter equation and invariants of links*, Invent. Math. 92 (1988), no. 3, 527–553.